Parametric & Nonparametric Models for Tests of Association

• Models we will consider
• \(X^2\) Tests for qualitative variables
• Parametric tests
  • Pearson’s correlation
• Nonparametric tests
  • Spearman’s rank order correlation (Rho)
  • Kendal’s Tau

Statistics We Will Consider

<table>
<thead>
<tr>
<th>DV</th>
<th>Categorical</th>
<th>Parametric</th>
<th>Nonparametric</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>mode, #cats</td>
<td>Interval/ND</td>
<td>Ordinal/~ND</td>
</tr>
<tr>
<td>univariate stats</td>
<td>mean, std</td>
<td>median, IQR</td>
<td></td>
</tr>
<tr>
<td>univariate tests</td>
<td>(X^2)</td>
<td>1-grp t-test</td>
<td>1-grp Mdn test</td>
</tr>
<tr>
<td>association</td>
<td>(X^2)</td>
<td>Pearson’s r</td>
<td>Spearman’s r</td>
</tr>
</tbody>
</table>

2 bg  
- \(X^2\)  
- t- / F-test  
- M-W  K-W  Mdn

k bg  
- \(X^2\)  
- F-test  
- K-W  Mdn

2wg  
- McNem  Crn’s  
- t- / F-test  
- Wil’s  Fried’s

k wg  
- Crn’s  
- F-test  
- Fried’s

M-W -- Mann-Whitney U-Test  
Wil’s -- Wilcoxin’s Test  
Fried’s -- Friedman’s F-test  
K-W -- Kruskal-Wallis Test  
Mdn -- Median Test  
McNem -- McNemar’s \(X^2\)  
Crn’s -- Cochran’s Test

Statistical Tests of Association w/ qualitative variables

Pearson’s \(X^2\)

\[
X^2 = \sum \frac{(o - e)^2}{e}
\]

Can be 2x2, 2xk or kxk – depending upon the number of categories of each qualitative variable

• H0: There is no pattern of relationship between the two qualitative variables.
• degrees of freedom \(df = (\#\text{columns} - 1) \times (\#\text{rows} - 1)\)
• Range of values \(0 \text{ to } \infty\)
• Reject H0: If \(X^2_{\text{obtained}} > X^2_{\text{critical}}\)
The expected frequency for each cell is computed assuming that the H0: is true — that there is no relationship between the row and column variables. If so, the frequency of each cell can be computed from the frequency of the associated rows & columns.

\[
f = \frac{\text{Row total} \times \text{Column total}}{N}
\]

\[
\begin{array}{c|cc|c}
   & \text{Col 1} & \text{Col 2} & \text{Row Total} \\
\hline
\text{Row 1} & 22 & 54 & 76  \\
\text{Row 2} & 46 & 32 & 78  \\
\hline
\end{array}
\]

\[
\begin{array}{c|cc|c}
   & \text{Col 1} & \text{Col 2} & \text{Row Total} \\
\hline
\text{Row 1} & \frac{(76*68)}{154} & \frac{(76*86)}{154} & 76  \\
\text{Row 2} & \frac{(78*68)}{154} & \frac{(78*86)}{154} & 78  \\
\hline
\end{array}
\]

\[
X^2 = \sum \frac{(o - e)^2}{e} \\
df = (2-1) * (2-1) = 1
\]

\[
X^2_{1, .05} = 3.84
\]
\[
X^2_{1, .01} = 6.63
\]
\[
p = .0002 \text{ using online p-value calculator}
\]

So, we would reject H0: and conclude that there is a pattern of relationship between the variables.

**Pearson's correlation**

- H0: No linear relationship between the variables, in the population represented by the sample.
- degrees of freedom \(df = N - 2\)
- range of values \(-1.00\) to \(1.00\)
- reject Ho: If \(|r_{obtained}| > r_{critical}\)

Pearson's correlation is an index of the direction and extent of the linear relationship between the variables.

It is important to separate the statements...

- there is no linear relationship between the variables
- there is no relationship between the variables
- correlation only addresses the former!
Correlation can not differentiate between the two bivariate distributions shown below – both have no linear relationship.

One of many formulas for $r$ is shown on the right.

- each person’s “$X$” & “$Y$” scores are converted to Z-scores ($M=0$ & $Std=1$).
- $r$ is calculated as the average Z-score cross product.

+$r$ results when most of the cross products are positive (both $Z$s + or both $Z$s -)

$-r$ results when most of the cross products are negative (one $Z$ + & other $Z$-)

Nonparametric tests of Association using ~ND/~Int variables

Spearman’s Correlation

- H0: No rank order relationship between the variables, in the population represented by the sample.
- degrees of freedom $df = N - 2$
- range of values -1.00 to 1.00
- reject Ho: If $|r_{obtained}| > r_{critical}$

Computing Spearman’s $r$

One way to compute Spearman’s correlation is to convert $X$ & $Z$ values to ranks, and then correlate the ranks using Pearson’s correlation formula, applying it to the ranked data. This demonstrates…

- rank data are “better behaved” (i.e., more interval & more ND) than value data
- Spearman’s looks at whether or not there is a linear relationship between the ranks of the two variables

The most common formula for Spearman’s Rho is shown on the right.

$$r = 1 - \frac{6\sum d^2}{n(n^2 - 1)}$$

To apply the formula, first convert values to ranks.

| # practices | # correct | rank | rank | d | $d^2$
<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>6</td>
<td>5</td>
<td>5</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>S2</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>-3</td>
<td>9</td>
</tr>
<tr>
<td>S3</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>S4</td>
<td>9</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>S5</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

$$\sum d^2 = 13$$

$$r = 1 - \frac{6 \times 13}{5 \times 24} = 1 - .65 = .35$$

For small samples ($n < 20$) $r$ is compared to $r_{critical}$ from tables. For larger samples, $r$ is transformed into $t$ for NHSTesting.

Remember to express results in terms of the direction and extent of rank order relationship!
So, how does this strange-looking formula work? Especially the "6"???

Remember that we're working with "rank order agreement" across variable — a much simpler thing than "linear relationship" because there are a finite number of rank order pairings possible!

If there is complete rank order agreement between the variables ... 
then, $d = 0$ for each case & $\Sigma d^2 = 0$
so, $r = 1 - 0$
$\Rightarrow r = 1 \rightarrow$ indicating a perfect rank-order correlation

If the rank order of the two variables is exactly reversed... 
$\Rightarrow \Sigma d^2$ can be shown to be $n(n^2 - 1)/3$
$\Rightarrow$ the equation numerator becomes $6 \cdot n(n^2 - 1)/3 = 2 \cdot n(n^2 - 1)$
so, $r = 1 - 2$
$\Rightarrow r = -1 \rightarrow$ indicating a perfect reverse rank order correlation

If there is no rank order agreement of the two variables ... 
$\Rightarrow \Sigma d^2$ can be shown to be $n(n^2 - 1)/6$
$\Rightarrow$ the equation numerator becomes $6 \cdot n(n^2 - 1)/6 = n(n^2 - 1)$
so, $r = 1 - 1$
$\Rightarrow r = 0 \rightarrow$ indicating no rank order correlation

The most common formula for Kendall's Tau is shown on the right.**

<table>
<thead>
<tr>
<th># practices</th>
<th># correct</th>
<th>rank practices</th>
<th>rank correct</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>6</td>
<td>21</td>
<td>4</td>
</tr>
<tr>
<td>S2</td>
<td>2</td>
<td>18</td>
<td>1</td>
</tr>
<tr>
<td>S3</td>
<td>4</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>S4</td>
<td>9</td>
<td>15</td>
<td>5</td>
</tr>
<tr>
<td>S5</td>
<td>5</td>
<td>10</td>
<td>3</td>
</tr>
</tbody>
</table>

**There are other formulas for tau that are used when there are tied ranks.

Nonparametric tests of Association using ~ND/~Int variables

Kendall's Tau

- $H_0$: No rank order concordance between the variables, in the population represented by the sample.
- degrees of freedom $df = N - 2$
- range of values -1.00 to 1.00
- reject $H_0$: If $| r_{obtained} | > r_{critical}$

All three correlations have the same mathematical range (-1, 1).
But each has an importantly different interpretation.

Pearson's correlation
- direction and extent of the linear relationship between the variables

Spearman's correlation
- direction and extent of the rank order relationship between the variables

Kendall's tau
- direction and proportion of concordant & discordant pairs
<table>
<thead>
<tr>
<th># practices</th>
<th># correct</th>
<th>rank practices</th>
<th># correct</th>
<th>rank</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>Y</td>
<td>X</td>
<td>Y</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S2</td>
<td>2</td>
<td>18</td>
<td>1</td>
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<td>2</td>
<td>1</td>
<td>3</td>
<td>0</td>
</tr>
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</tr>
</tbody>
</table>

For each case...

C = the number of cases listed below it that have a larger Y rank
(e.g., for S2, C=1 → there is one case below it with a higher rank - S1)

D = the number of cases listed below it that have a smaller Y rank
(e.g., for S2, D=3 → there are 3 cases below it with a lower rank - S3 S5 S4)

\[
\tau = \frac{2(C-D)}{n(n -1)} = \frac{2(6 - 4)}{5(5 - 1)} = \frac{4}{20} = .20
\]

For small samples (n < 20) \(\tau\) is compared to \(\tau\)-critical from tables.
For larger samples, \(\tau\) is transformed into Z for NHSTesting.